Introducing algebra with interactive geometry software

Kate Mackrell
kmackrell@ioe.ac.uk
Institute of Education
University of London
U.K.

Abstract: *In order to consider the ways in which interactive geometry software might facilitate the expression of generality, an exploratory inspection using a series of tasks related to finding a connection between the radius and area of a circle was performed using Cabri II Plus, Casyopée, Cinderella, GeoGebra, Geometer's Sketchpad and TI-Nspire (CAS). It was found that the programs specifically designed to include algebraic functionality had less potential for enabling students to develop the concept of variable and to generate symbolic representations between variables than did the programs with less overt algebraic functionality. An implication is that more attention needs to be paid to the needs of learners in the early stages of algebraic thinking when incorporating algebra within IGS.*

1. Introduction

There is increasing interest in the idea of combining interactive geometry software (IGS) and computer algebra software (CAS). Early IGS programs, Cabri Geometre [9], Geometer's Sketchpad [5], and Cinderella [16] have been further developed to include algebraic capabilities such as algebraically defined graphs. Programs have also been developed that explicitly aim to combine geometry and algebra, such as GeoGebra [4] , Casyopée [10] and TI-Nspire CAS [18].

"Algebra is … the language of generalisation of quantity. It provides experience of, and a language for, expressing generality, manipulating generality, and reasoning about generality." [13] This definition will be used to analyse the ways in which such IGS environments have the potential to contribute to the learning of algebra.

An algebra learning environment should:

a. Provide experience of generality. Such an environment needs to contain objects about which generalizations can be made. A recent review of software currently used for teaching algebra [19] shows that the objects commonly provided are algebraic objects such as functions, represented symbolically and graphically. Most current IGS programs also have the functionality to represent and explore functions (see [10] for some of the possibilities for exploration in an IGS).

However, very few environments provide objects which are appropriate for learners who have yet to understand the concept of a variable.   Grid Algebra [3], in which objects are related sequences of numbers, and the MiGen project [14] in which objects are visual patterns made with tiles, are exceptions. These programs were specifically designed to introduce the concept of a variable and scaffold learners in generating symbolic expressions of the relationships between variables.

IGS also provides appropriate objects; geometric objects such as a circle may be measured to give dynamic numbers [6] representing its radius, area, or circumference. These numbers change as the size of the circle is changed, mediating the concept of variation and ultimately of variable. One advantage to IGS objects is that they can be drawn from meaningful contexts, either curriculum

mathematics, problems such as modeling the area of an enclosure created by using a rope of fixed length for three sides [10], or student-created structures. A second advantage is that, rather than attempting to find a pattern connecting a series of objects, a single object may be manipulated, which may change in either a discrete or continuous manner. An example is shown in Figure 1 below.
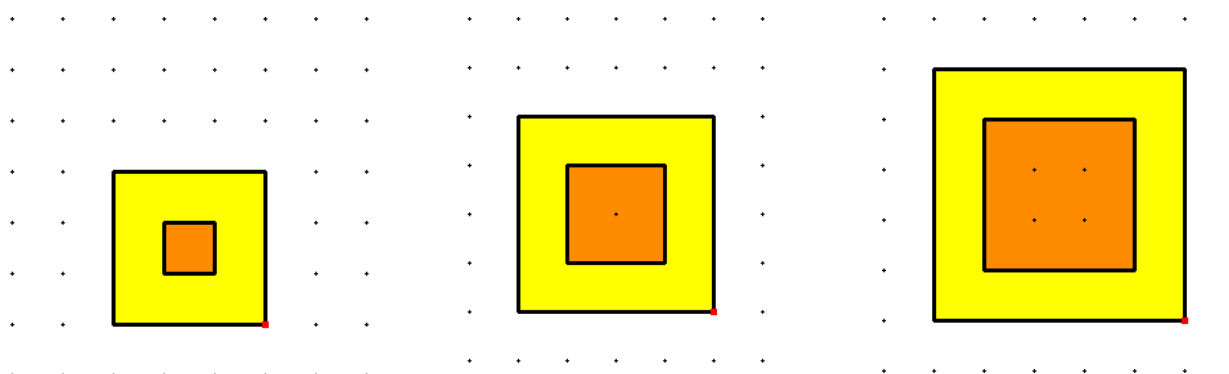

Figure 1 A series of objects representing a pattern

In a typical task involving static diagrams or materials, the student might be asked to find the fourth pattern in the series and then to find a relationship between the length of the side and the orange and yellow areas. Typical issues are that students attend to numbers generated, and in particular to the ways in which these numbers change between different objects in the sequence rather than to the structure of the pattern itself [15]. However, using interactive geometry, the student may be presented with one such object. The screenshots above are of one Cabri object, dragged to change the length of the base square.   See files *discrete_pattern.fig* and *continuous_pattern.fig* for discretely changing and continuously changing examples of this object.

b. Enable generalizations to be expressed and validated.   In the early stages of algebraic thinking, this involves finding relationships, ultimately expressed in symbols, between numbers at different positions on a grid [3] or between pattern number and number of tiles [14], or between radius and area of a circle, and testing whether the generalizations are valid. The process of finding a generalization might involve reasoning about the structure of a situation or, in IGS, representing the situation in a table or a graph. The means by which geometric constructions can enhance student awareness of dependence and independence [7], student understanding of the concept of function [2] and the way in which geometric objects and graphs can be linked [8] suggest that IGS might play an important role in this area.

c. Enable the manipulation of generalizations. This includes traditional algebraic manipulation such as rearranging formulae, solving equations, etc. and, along with graphical representation, is the focus of much of the available algebra teaching software [19]. Of the IGS listed, Casyopée, GeoGebra, and TI-Nspire have the capability for direct algebraic manipulation.

d. Enable reasoning about generality. Algebraic manipulation might be necessary for formal proof. However, IGS is an excellent environment for the development of visual "proofs", such as the circle that unfolds into an approximate parallelogram shown below. [1]
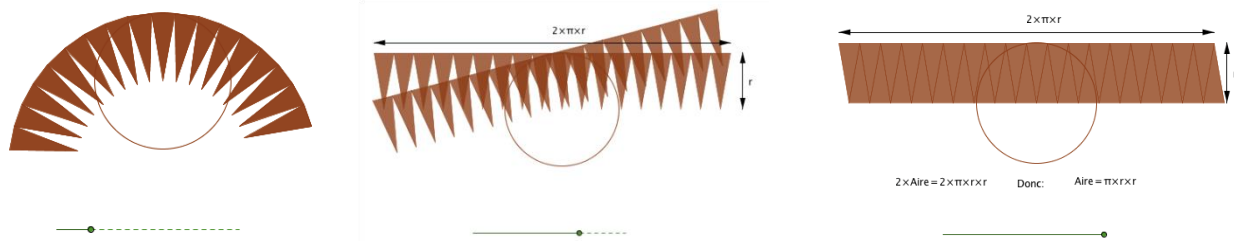
Figure 2 Visual "proof" of the connection between the radius and area of a circle

This paper reports on an exploratory inspection by the researcher of the six IGS programs above in order to identify the functionality in each program that would specifically facilitate or impede the expression of generality, with focus on the earliest stages in algebraic thinking: developing the concept of a variable, associating it with an algebraic symbol, and making a connection between two such variables.

The versions of the software used are as follows: Cabri II Plus version 1.4.5, Casyopée version 2.6.5, Geometer's Sketchpad version 5.03, Cinderella 2.6 build 1585, GeoGebra version 4.0, and TI-Nspire CAS Teacher Software version 3.0.1.1753.

2. Method

A series of tasks was performed with each program with the aim of finding and verifying a formula for the area of a circle. The topic, finding the area of a circle, is universal, and each task involved processes that would also apply to other mathematical explorations. The pedagogical approach (involving exploring and gathering information about a mathematical situation, making and testing conjectures, then generalizing and proving results) has been promoted extensively in the UK since the 1980's.

3. Results

The files obtained may be downloaded using the links given for each task[1]. If files do not open properly from the link, right-click on the link, save the file, open the appropriate software and then open the file from within this software[2]. Download all files for: Cabri Casyopée Cinderella GeoGebra GSP TI-Nspire

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[1] File extensions are .fig (Cabri), .CAS (Casyopée – note that the program generates a text file as well. Individual Casyopée links hence will download the two files, zipped), .cdy (Cinderella), .ggb (GeoGebra), .gsp (GSP) and .tns (TI-Nspire). GSP allows multiple pages within a document, and hence each GSP file is the same, apart from the page that shows when the document is opened. TI-Nspire also allows multiple pages, but does not allow existing pages to be copied in order to be further developed within the same document and hence separate files were created.

[2] A particular problem is that Matlab uses the same file extension as Cabri II Plus: if you have Matlab on your computer you will need to do this with all Cabri files.

Task 1: Modelling a situation
download: all Cabri Casyopée Cinderella GeoGebra GSP TI-Nspire

In each program, a circle was created given a centre point and a point on its circumference.   The area and radius of the circle were then measured. See [12] for some implications for number and algebra of the software design decisions involved in the processes of creating and measuring. Screenshots of the figures constructed are shown below.
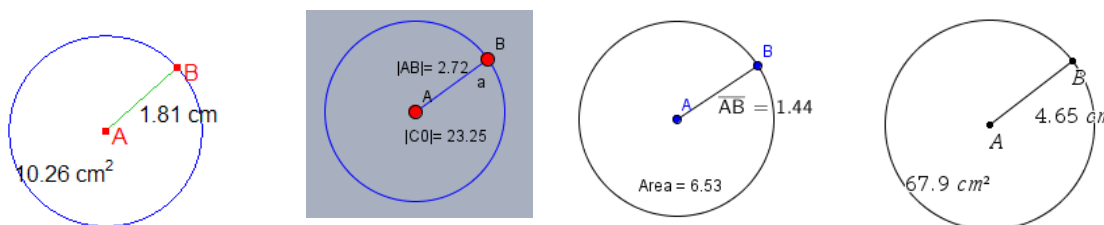
Figure 3 The measured circle in Cabri II Plus, Cinderella, GeoGebra, and TI-Nspire
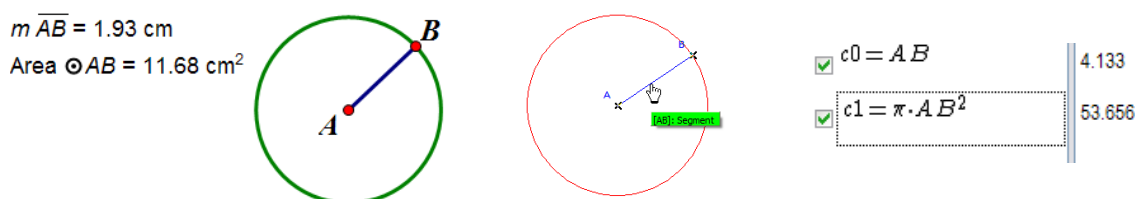

Figure 4 The measured circle in Geometer's Sketchpad and Casyopée

A central issue in using geometry to introduce symbolic algebra is immediately apparent: letters are used to label geometric objects such as points and also variables (the radius and area of the circle). The way in which measurements are named and annotated is hence important.

In Cabri and TI-Nspire, measurements are not named by default. Watching measurements vary (particularly when associated with the objects measured through the use of units) may give students a concept of variation without having to immediately associate this with a named variable: naming (possible in each) may be a next step.

In the other programs, measurements are labelled automatically as the measurement is made. This has the advantage that the labels can give the measurement an immediate meaning. However, conventions such as AB, |AB|, or $\overline{AB}$ for the length of the segment need to be interpreted, and only GSP consistently names objects clearly. Casyopée chooses arbitrary variable names for measurements, with no relation to the name of the object measured. In GeoGebra a measurement is identified as area, but not of the circle specifically. The name C0 used by Cinderella is not visible as the name of the circle and |C0| does not obviously represent the area rather than the circumference of the circle. However, the ability to change or hide the label (in all but Casyopée) means that the user could, for example, change AB to "radius of circle c" to clarify or to "r" as they develop the ability to concisely name a variable.

Casyopée has a particular issue with measurement. In the other programs, measurement happens by clicking to select the appropriate tool or command and then clicking to select the objects required for the measurement. An example with Cinderella is shown below.
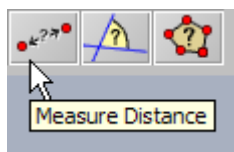

Click to select the tool
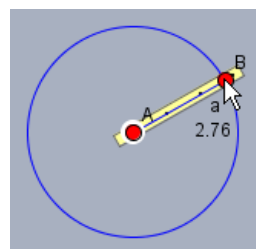
Click on A, hold down and drag
to B to select distance AB.

Figure 5 Measuring the distance between two points using Cinderella

In these programs, the measurement tools are "black boxes": the student is given no information about how measurements are made, and hence are able to meaningfully look for relationships between them. In contrast, in Casyopée, measurements are created by specifying a calculation: typing in AB (or d(A, B))will for example calculate the distance from A to B, and the area of the circle needs to be calculated using the length of the segment. The details of this calculation are then displayed, making any further search for a relationship between the two measurements somewhat meaningless.
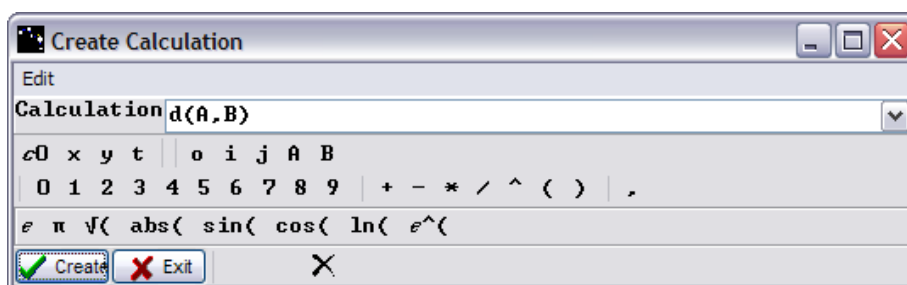
Figure 6 "Measuring" the distance between two points using Casyopée
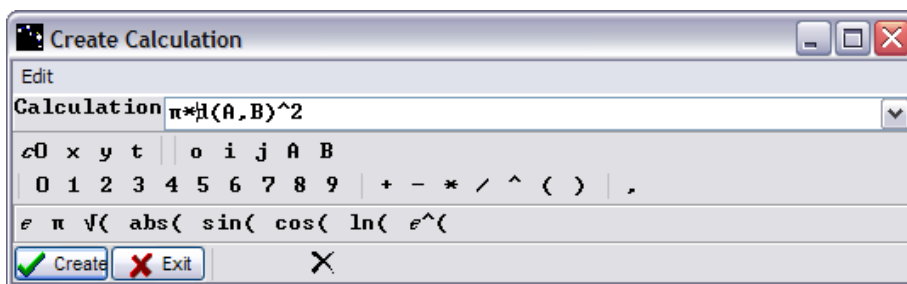
Figure 7 "Measuring" the area of a circle using Casyopée

The next step is to change the radius of the circle, observe the effect on its area and make a conjecture about the relationship between the two. The visual changes in the circle are linked with the changes in the numbers measuring the circle. A question is whether the proximity of the measurement to the object measured has any effect on user understanding of this link. All measurements are attached to their objects by default in Cabri, Cinderella and TI-Nspire, can be attached in GSP, but are displayed in a separate window in Casyopée. GeoGebra is inconsistent: the measurement of the radius is attached to the segment, and can be detached, but the measurement of the area is not and cannot be attached without further construction.

As the radius increases, the area increases, so a first conjecture might be that the area is some multiple of the radius.

Task 2: Test the conjecture by calculating area/radius
download: all Cabri Casyopée Cinderella GeoGebra GSP TI-Nspire(expression) TI-Nspire(calc)

This method of testing the conjecture will be used to illustrate a potential connection between calculation and algebraic understanding: that the arithmetic operation of dividing one number by another may be generalized to the algebraic operation of dividing one variable by another.

The ability to act as a mediator between numerical and algebraic operations depends upon a number of software design features. The three examples below exemplify these.
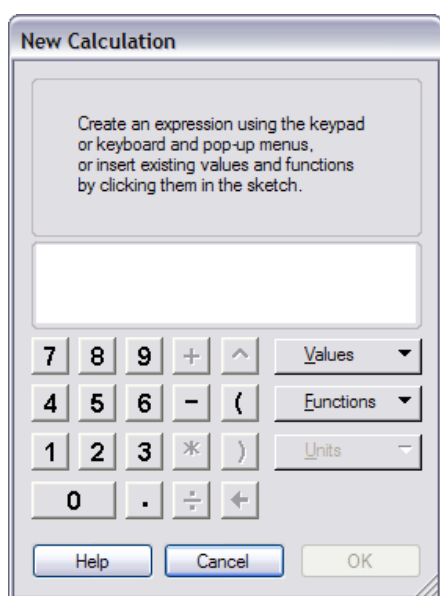
GSP Calculator (similar to Cabri and Casyopée)

GeoGebra input bar (similar to Cinderella)

$$\frac{a}{b}$$

TI expression created using Text tool (also possible with Cabri)

Figure 8 Different "calculators"

The first feature is the type of tool used to perform calculations. A clear link is made with arithmetic operations in Cabri, Casyopée and GSP by specifically designating the tool as a

calculator and giving it the appearance of a calculator. Operations may be entered either by typing or by clicking on an icon.

In contrast, this link is not made apparent by the use of the input bar in GeoGebra and the function-defining tool with icon f(x) in Cinderella. All operations must be entered by typing, and the user is required to know symbols for multiplying, "*", dividing, "/", and exponentiation, "^", which differ from those used on paper or with a calculator.

The TI-Nspire demands the most algebraic awareness: either the radius and area both need to be specifically defined as variables and then operated upon in a calculator window, or an algebraic expression (as shown above) needs to be set up into which values may be substituted.

The major consideration, however, is the way in which dynamic numbers, or numbers which represent variables, are entered and represented in calculations. The simplest method to enter a dynamic number, used by GSP, Cabri, and Cinderella, is to click on the number on the screen. Crucially, the user does not need to recognize that the number represents a variable. Casyopée and TI-Nspire also allow numbers to be entered by clicking, but here the numbers need to be recognized as variables. Casyopée gives the names of all possible variables in its calculator and a number is entered by clicking on the name of its associated variable. In TI-Nspire, numbers can be substituted into an expression by clicking on their value on the screen. (Alternatively, variable names, either typed or chosen from a dropdown list, may be entered in a calculator window).

Entry of dynamic numbers in GeoGebra is particularly problematic as the input bar will only accept text entered by typing. This means that dynamic numbers must be represented symbolically. A further problem is that, when created, measurements are assigned specific names by the software – and these names are not necessarily shown in the geometry window. It is necessary to open the algebra window, which gives potentially confusing information such as the equation of the circle, in order to find the appropriate names to enter. Text entry is also prone to syntax errors.
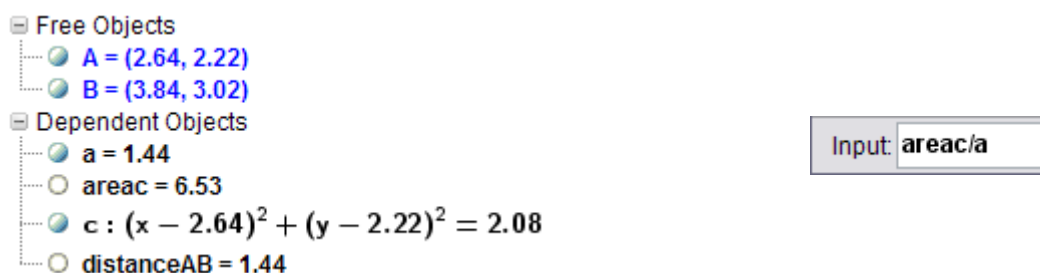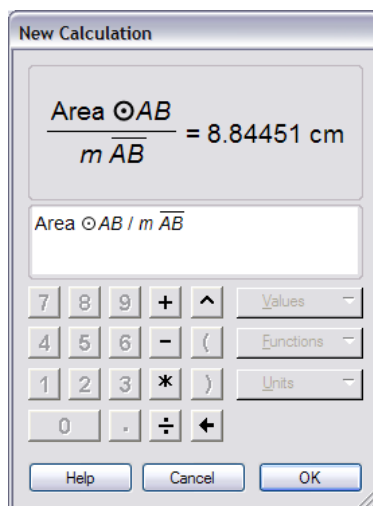

Figure 9 GeoGebra Algebra window and Calculation

The algebra window above also highlights a fundamental mathematical error in GeoGebra: the name of a geometric object is used as a variable and assigned a value. The construction protocol below confirms that the letter a is used as the name of segment AB and is also assigned the value of the length of AB. For further discussion of this issue see [11].

| No.	Name	Definition	Value
1	Point A		A = (2.64, 2.22)
2	Point B		B = (3.84, 3.02)
3	Circle c	Circle through B with center A	c: (x - 2.64)² + (y - 2.22...
4	Segment a	Segment [A, B]	a = 1.44
5	Number dist...	Distance of A and B	distanceAB = 1.44

Figure 10 Geogebra Construction Protocol

Further differences may be seen in the way in which dynamic numbers are represented in the calculator. In Cabri, GSP, and Cinderella, numbers are named, using the label on the page (GSP), the name of the object generated internally when the object was created (Cinderella), or arbitrary letters (Cabri, and GSP for unlabelled measurements). This may be a powerful means of facilitating the recognition that dynamic numbers can be represented by variables. GSP is particularly clear, as shown below:

New Calculation

$$\frac{\text{Area } \odot AB}{m\,\overline{AB}} = 8.84451 \text{ cm}$$

Area $\odot AB$ / m $\overline{AB}$

7	8	9	+	^	Values
4	5	6	−	(	Functions
1	2	3	*	)	Units
0	.	÷	←		

Help   Cancel   OK

Figure 11 GSP Calculation

In Cinderella in contrast, clicking on the value of |C0| (the area of the circle) and the value of |AB| (the radius of the circle) enters the following in the calculator:

Enter a CindyScript line

A0/dist0

Function Text:
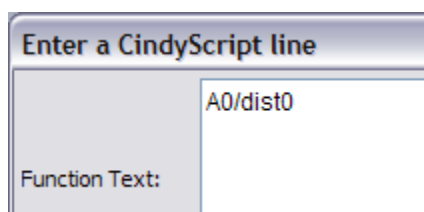
Figure 12 Cinderella Calculation

These are the names assigned internally to the area and distance when they are created, but, as they do not match the label on the screen, they may be problematic for the user.

There are also differences in the way the results of the calculation are displayed. For Cabri, Cinderella, GSP and TI-Nspire the results are shown on the geometry page, as shown in the examples below.
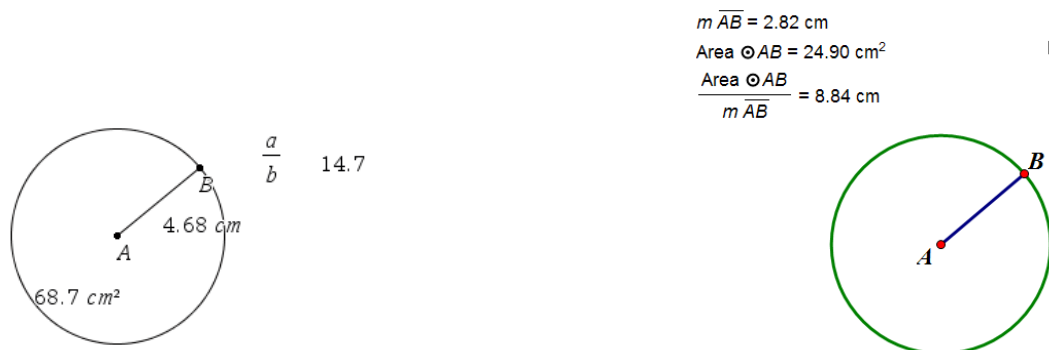
Figure 13 Calculation Results for TI-Nspire and GSP

Calculations are given either without details (Cabri and TI-Nspire), with details (GSP), or as chosen by the user (Cinderella). All calculations may be given different labels.

In this situation it is easy to see that the result changes as the radius changes: the conjecture concerning a linear relationship was incorrect. This is not as clear with Casyopée and GeoGebra, however, as the result is assigned an arbitrary variable name and displayed in a different window, as shown below.
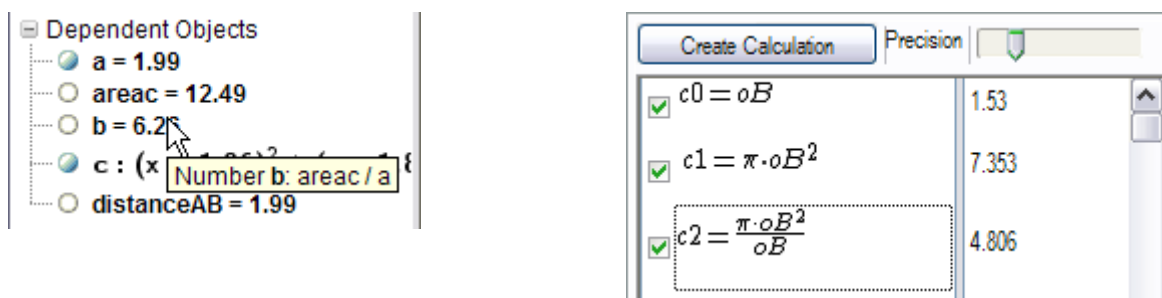
Figure 14 Calculation Results for Casyopée and GeoGebra

It is relatively straightforward to find the new variable in Casyopée, as it shows the details of the calculation (and optionally its value), and calculations are listed in order of occurrence. However, as GeoGebra's algebra window is organized alphabetically and the new variable is only shown with its value, it can be difficult to recognize, although, as shown above, a description will appear briefly when the cursor runs over the value. It is possible, however, to relabel the measurement and display it on the geometry screen.

Task 3 Make a table of data and plot the data
download: all Cabri(table) Excel graph of Cabri data Casyopée Cinderella(table) Cinderella(plot) GeoGebra GSP TI-Nspire(table) TI-Nspire(plot)

The danger of attempting to find a pattern from numbers rather than from the structure of a situation has been stressed [14] [15]. An activity such as constructing a circumscribed and inscribed square and comparing the areas of these squares and the circle might lead students to realize from the structure of the situation that the area of the circle must be proportional to the area of the squares, and hence to the square of its radius, with constant of proportionality between two and four.
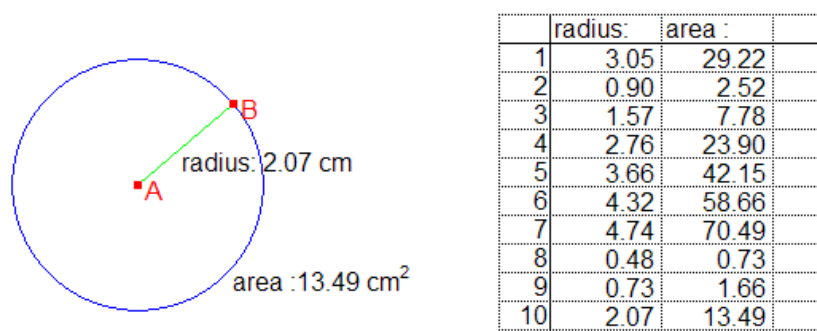
However, in situations where the structure is not immediately apparent, particularly when the relationship is non-linear, or involves continuous or random variables, gathering and plotting data might give useful structural insight. Gathering data at irregular intervals also avoids the problem of sequential data, where patterns are found connecting adjacent terms rather than connecting variables as a whole. Data collection also emphasizes the idea of variation by giving a record of specific values that the variables have had at one time.

All of the programs allow tables to be made in which measurement values may be recorded. In Cabri II Plus and GSP, creating the table requires little algebraic awareness: choose the table tool and click to select the measurements or calculations to be tabulated. If measurements have names, then these names are used as column headings.
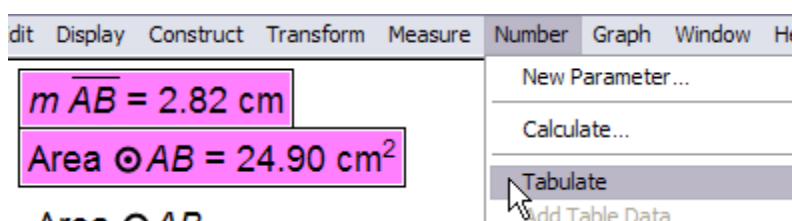
Figure 15 Creating a table with GSP

For the Cabri example below, data was collected by dragging point P and pressing the TAB key.

	radius:	area :
1	3.05	29.22
2	0.90	2.52
3	1.57	7.78
4	2.76	23.90
5	3.66	42.15
6	4.32	58.66
7	4.74	70.49
8	0.48	0.73
9	0.73	1.66
10	2.07	13.49

Figure 16 Cabri data table

The other programs require a specific awareness of the measurements as algebraically represented variables and have a more complicated collection technique. In Casyopée, the figure must first be reconstructed so that the circle is centred at the origin and has radius point on an axis, then a model must be set up between the two calculations representing radius and area, and then this model must be displayed as a table (note that neither the model nor the table can be saved to file). The values closest to those given by the current configuration are highlighted.
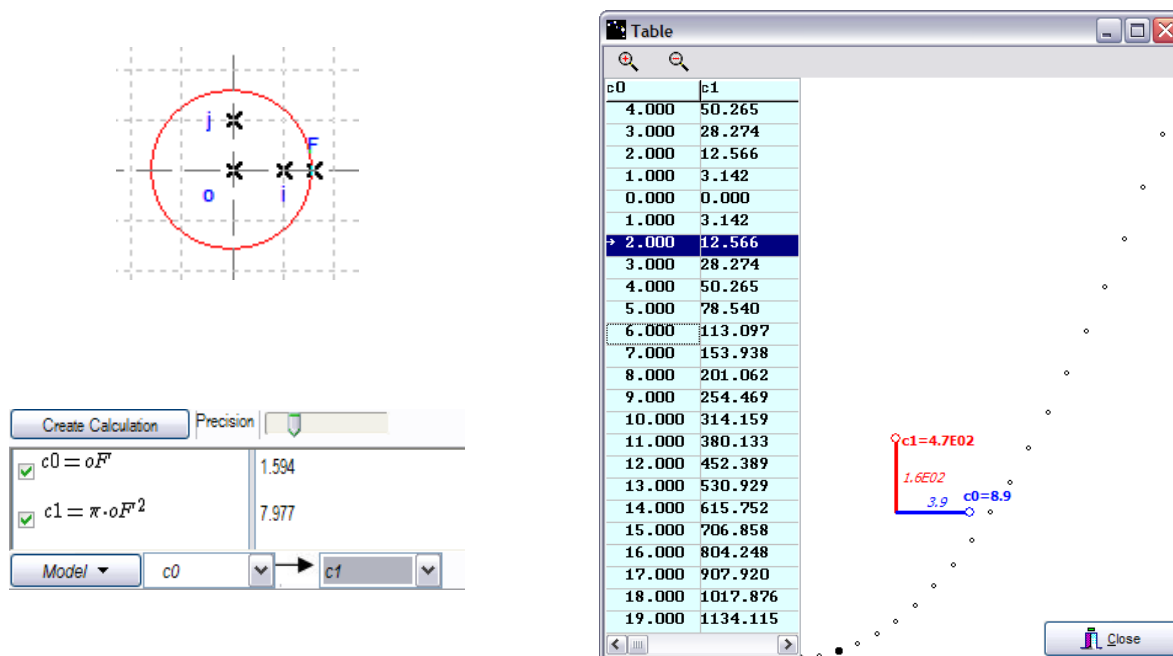

Figure 17 Creating a data table with Casyopée

TI-Nspire and GeoGebra both enter data into a spreadsheet view, which has the advantage that further calculations can be performed on the data. In TI-Nspire, a measurement must first be explicitly defined as a variable, and then a spreadsheet column must be defined as a capture of this variable. In GeoGebra, data needed to first be graphically represented by defining (and hence plotting) a point with coordinates given by the variables for radius and area. The coordinates of the point may then be recorded in a spreadsheet.

In Cinderella, the inbuilt programming language Cindyscript must be used to both gather data and represent such data graphically, which, for a beginner, might be difficult[3]. However, there have been a number of arguments for the value of programming in introducing algebraic ideas [17]: the lack of easy data-collection functionality does not preclude Cinderella from offering significant possibilities for the development of algebraic thinking.

GSP allows the data from the table to be easily graphed within the geometry page. Axes are constructed with an appropriate scale and all data is displayed.

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[3] See the Cinderella files for this task for instructions on how to do this.

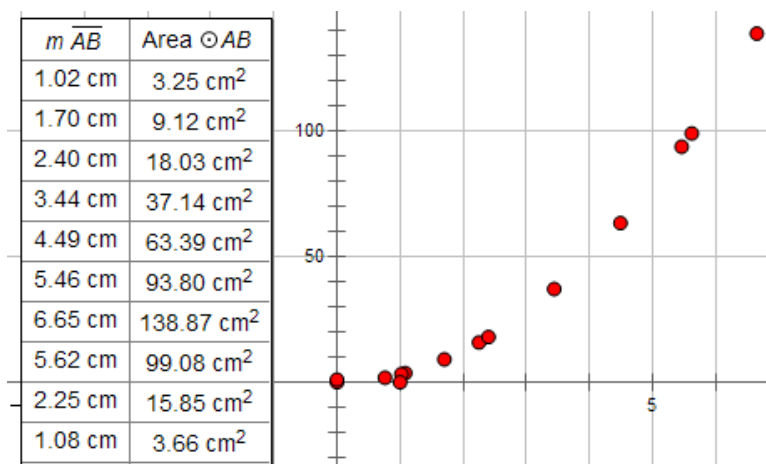| $m\,\overline{AB}$	Area $\odot AB$
1.02 cm	3.25 cm²
1.70 cm	9.12 cm²
2.40 cm	18.03 cm²
3.44 cm	37.14 cm²
4.49 cm	63.39 cm²
5.46 cm	93.80 cm²
6.65 cm	138.87 cm²
5.62 cm	99.08 cm²
2.25 cm	15.85 cm²
1.08 cm	3.66 cm²

Figure 18 GSP table with plotted data

Cabri II Plus allows the data to be copied and pasted into a spreadsheet (see file *3_plotCabridata.xls* in the folder for this section), from whence it can be graphed and further analyzed, but it does not allow this to take place on the page itself. In Geogebra, points are plotted by selecting the data and turning it into a list. Casyopée automatically showed a graph alongside the table data, with the point representing the highlighted values clearly visible. As shown below, TI-Nspire would only plot the data within a new window, rather than within the existing graphs and geometry window.
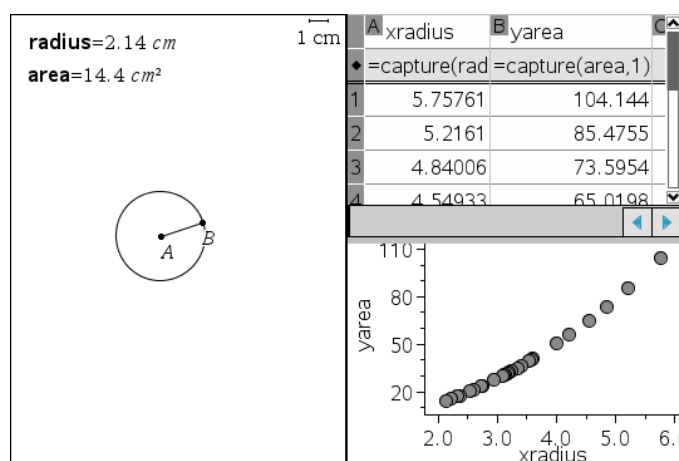
Figure 19 TI-Nspire table with plotted data

Task 4: Use the measurements to create a graph of area against radius
download: all Cabri(point+trace) Casyopée(point+trace) Cinderella(point+trace) GeoGebra(point+trace) GSP(point+trace) TI-Nspire(point+trace) Cabri(locus) Casyopée(locus) Cinderella(locus) GeoGebra(locus) GSP(locus) TI-Nspire(locus) Cabri(changing axes) Cinderella(changing axes) GeoGebra(changing axes) GSP(changing axes) TI-Nspire(changing axes)

One of the most powerful features of IGS programs is the ability to visually represent the way in which measurements vary: a graph of area against radius may be constructed directly from the

existing measurements without needing to either collect and plot data or to define an algebraic relationship.

It is possible in all programs to show inbuilt coordinate axes and then directly plot the point representing (radius, area). GSP is the simplest; the point can be plotted by selecting numbers for x and y coordinates and then selecting the appropriate tool.
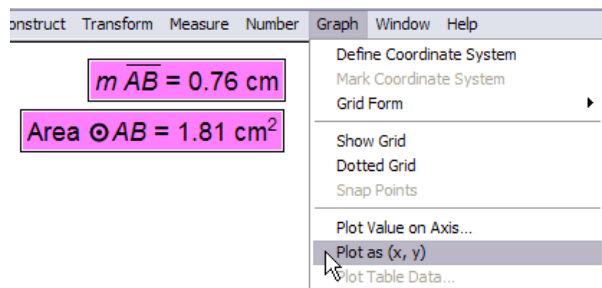
Figure 20 Plotting a point with GSP

Cabri II Plus and TI-Nspire require a geometric construction for which a custom tool may be created in Cabri II Plus. Each of the other programs required the coordinates of the points to be entered in a particular syntax. An example using Cinderella is shown below: a point must first be created and then have its coordinates set to particular values using the f(x) icon and clicking on the values on the page.
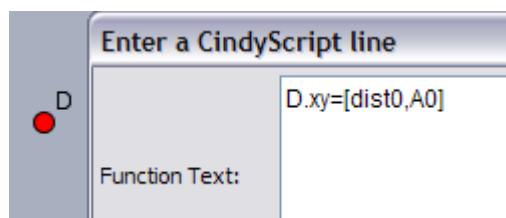
**Figure 21** Plotting a point with Cinderella

In GeoGebra (for which the required point had been plotted in order to gather data) plotting required identifying radius and area as specific variables. Casyopée required the coordinates of the point to be recalculated.

Each program enabled the point to be traced, to create a visual record of the way in which area varied as radius was changed. Note that many of the programs do not save traces: they may be recreated by dragging point B in each file.

It is possible to go further than just tracing the point by creating the locus of all such points as the radius point moves. This locus forms the graph of area against radius for the circle. In order to create this locus, it is necessary to redefine the radius point as a point on a path. This type of redefinition may be valuable in any situation where an independent variable needs to be restricted. Redefinition is not possible in Casyopée: but in order to generate and graph table data the figure was reconstructed to place the radius point on an axis, which acts as a path. In Cinderella, Cabri,

GSP and TI-Nspire[4] a segment may be constructed from the centre A to some arbitrary point C and the radius point B may easily be redefined to be a point on this segment. However, despite having the most extensive facilities for redefinition of all the programs in that not only the dependency of an object but also its basic geometric type can be changed, GeoGebra did not immediately allow this redefinition: apparently an "earlier" object point cannot be redefined to depend on a "later" object regardless of the geometrical independence of the later constructed object. The construction protocol had to be re-ordered to place the new segment above the radius point before the redefinition could take place.

It was then possible to create the locus of the point representing (radius, area) as the radius changed for each program. A problem that then arises, however, is that the area increases much more rapidly than the radius, so that the graph is only visible for small values of the radius and it is difficult to make any conjectures concerning its shape. This can be seen in figure 22 below.
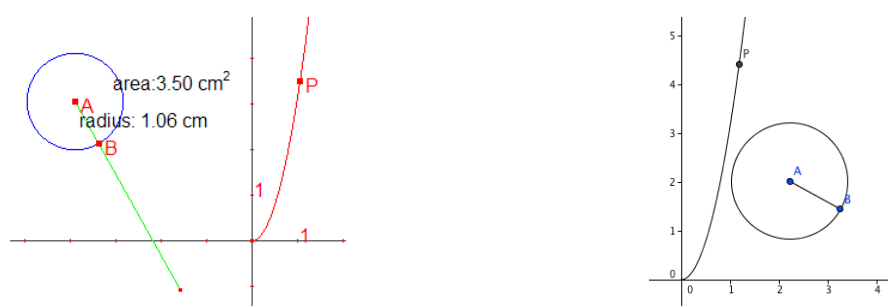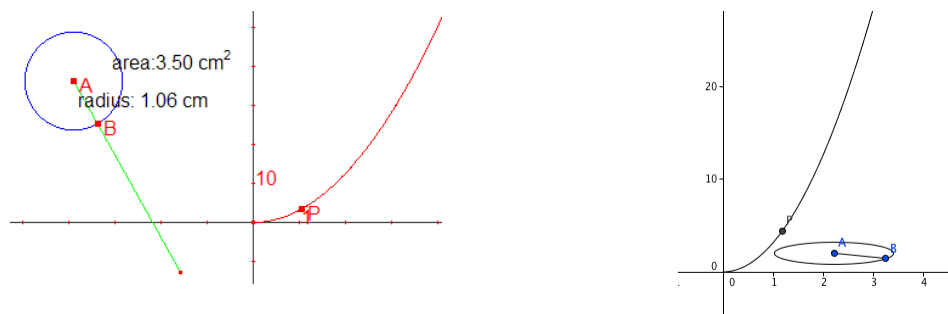

Figure 22 Cabri and GeoGebra circles and loci

It would be useful to show more of the graph by changing the scale on the y axis, but in Cinderella it was only possible to change the scale by zooming in or out, and in Casyopée it was necessary to export the calculation to a separate graph window in order to change the scale. This was possible in the other programs, however. In Cabri, TI-Nspire[5], and GSP, as the scale changes the circle remains the same, but the graph flattens as shown in the Cabri example below.

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[4] In TI-Nspire, the view must first be changed to Plane Geometry to allow this segment to be drawn and the redefinition to take place.

[5] For the TI-Nspire, if a circle is defined in a plane geometry view, then the circle is defined relative to the underlying plane and does not change if axes are shown and the scale on the axes is changed. If, in contrast, a circle is drawn in the graphing view, then this circle is defined relative to the coordinate axes of this view and will change shape and location if the axes are modified.

Figure 23 Cabri and GeoGebra circles and loci with unequal axis scaling

However, in Geogebra the shape of the circle changes as shown above; this is a consequence of the design decision to use a fundamentally algebraic representation of objects and to allow axes to be scaled independently. See [11] for further discussion of this issue. In this situation, and any situation in which an object of a specific shape is being modeled, the change of shape might be highly confusing to students.

Task 5: Find the equation for this curve
download: all Cabri(direct equation)
Cabri(function graph) Casyopée(function graph) Cinderella(function graph)
GeoGebra(function graph) GSP(function graph) TI-Nspire(function graph by expression)
TI-Nspire(function graph by equation)
Cabri(graph + slider) Casyopée(graph + parameter) Cinderella(graph + slider)
GeoGebra(graph + slider) GSP(graph + slider) TI-Nspire(graph + slider)

Using the Cabri coordinates and equation tool, it was possible to find the equation of the locus directly. This was not possible with any of the other programs: GeoGebra did not even list the locus as an object in the algebra window. TI-Nspire and GeoGebra allowed a quadratic regression to be done on the data collected, however, which indirectly results in the equation of the locus. Casyopée will derive a relationship between the variables representing radius and area, but, as this relationship needed to be used to calculate area initially, the derivation in this case is of little value. See [10] for a description of a situation in which the ability to derive relationships is valuable.

It may be more useful for students to model the locus by fitting to it the graph of an algebraically defined function. All programs enable functions to be graphed, but the differences in the way that this is achieved may impact student understanding. A fundamental requirement for this task is for the function, an algebraically defined object, to be visible in the same window as the locus or trace, which is not the case with Casyopée. A second issue is the extent to which the concept of function, generally found to be problematic to students, needs to be used; at the level of this particular task, students may be encountering a non-linear graph (or any algebraically defined graph) for the first time. Although in each program it is straightforward to define and graph the function $y = 2x^2$, for instance (based on the shape of the graph), all but Cabri and TI-Nspire require either the use of a specific function tool or some awareness of function notation. In Cabri and TI_Nspire, an expression is created that may be evaluated for particular numbers or applied to axes to create a graph of the related function. This may be a useful introduction to the idea of graphing. However, in Cabri specific function notation cannot be used, which may be a disadvantage for more

experienced learners. The number 2 could be modified either by changing the definition of the function or expression, or by introducing a parameter k to replace the 2, as shown below.
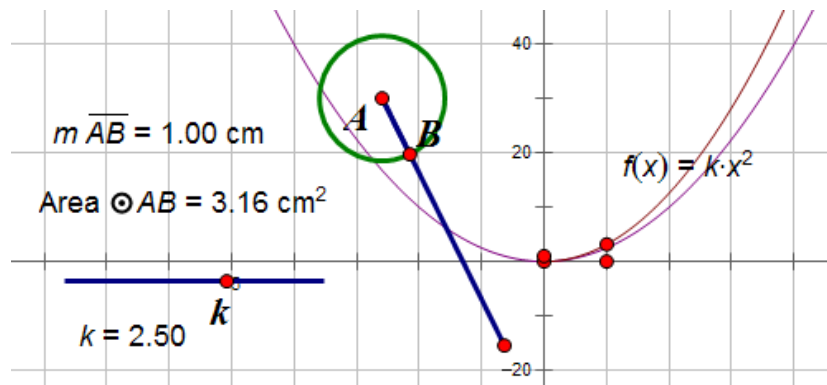

Figure 24 GSP function with slider

GeoGebra has a particularly simple means of introducing a parameter by means of a slider object: sliders in the other programs must be constructed geometrically (apart from Casyopée, where a parameter k may be used, but cannot be connected to a slider). By manipulating the parameter, students should derive an equation such as $y=kx^2$, where k is approximately 3.14 for the best fit curve.

Task 6: Test the formula found
download: all Cabri Casyopée Cinderella GeoGebra GSP TI-Nspire

One way to test this formula would be to edit the calculation of area/radius to area/radius^2, which might suggest that a value for k such as 3.13 is only accurate to one decimal place. It is also possible to create a function (GSP, Cinderella, GeoGebra, TI-Nspire, Casyopée) or expression (Cabri, TI-Nspire) giving $3.13*r^2$, substitute the radius for r and compare the result with the measured area as shown in the example below. Casyopée was problematic in that the function definition window could not be shown at the same time as the geometry window, and that the function result was given as a fraction and hence difficult to compare with the calculated result.
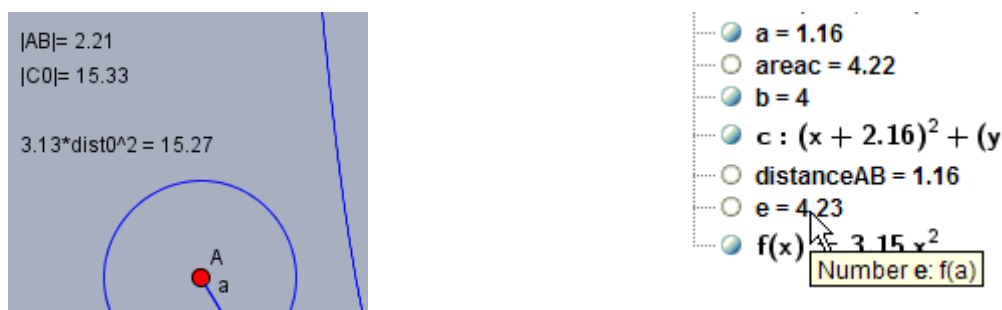

Figure 25 Testing the formula with Cinderella (k=3.13) and GeoGebra (k=3.15)

Task 7: Justify the formula found

This task, crucial pedagogically, is beyond the scope of this paper. However, there are a number of ways in which IGS might help to show students why the formula derived works. A comparison between the area of the circle and inscribed and circumscribed squares has already been suggested. Polygons of an increasing number of sides can also be inscribed and circumscribed. The visual "proof "in the introduction is another possibility.

4. Conclusion

The programs explored here differ in the functionality which might support the initial stages of learning to express generality. Cabri and Sketchpad have the greatest potential for scaffolding the development of algebraic awareness in the particular task shown. In particular, each allowed variables to be manipulated without the explicit concept of a variable, but also enabled the development of this concept. The other programs all demanded the use of variables without giving scope for the development of the concept. However, Cinderella, with its programming language, might afford a unique and entirely different route into the development of the concept of variable.

An issue with the three more overtly algebraic programs, Casyopée, GeoGebra and TI-Nspire is the existence of multiple windows and the consequent difficulty in both communicating between windows and in displaying objects where visually useful. The connections between number, algebra, and geometry essential to this task may be easier to establish when objects are not separated into different windows.

TI-Nspire is an attempt to synthesize geometric functionality, to a large extent derived from that of Cabri, with the functionality of a graphics calculator and a CAS system. It is possible that it lacks a calculator within the geometry window because it has a separate calculator window: however, numbers must be specifically defined as variables to be exported to this window and it loses the immediacy of the Cabri calculator. Note that it also incorporates a programming language, although this is less intrinsically connected with its geometry.

Casyopée and GeoGebra are more problematic. Casyopée does not have many of the functionalities required for this task. The use of the calculator for measurement is particularly problematic in that the relationship between radius and area needed to be used to "measure" the circle area: exploring the circle to discover this relationship consequently does not have much point. However, Casyopée started as a CAS program, designed to be used in the upper years of secondary school, and has only recently incorporated an IGS module. Additional functionality such as measurement, the ability to collect and plot data and the ability to redefine points and to create a locus could usefully be incorporated.

GeoGebra had the functionality required for the task and had some useful features, such as the ability to perform calculations with table data and to easily define a slider. However, there are a large number of problems in the way the functionality has been implemented. A major issue is the prevalence of text input, which requires the use of named variables, is open to syntax errors, and requires the algebra window to be open in order to locate variables, displaying a large amount of information irrelevant to the task. A serious mathematical mistake is made in using a geometric label as a variable, and confusion may be created by basing redefinition on order of construction

rather than actual dependence and by not allowing the shape of a graph to be manipulated separately from the model of an object. Minor issues include a lack of units, locus not listed in the algebra window, needing to plot a point to collect data and difficulty in placing the results of calculation in the geometry window. Unlike Casyopée, GeoGebra claims to be "from elementary school to university level". This, and its increasing popularity among teachers, make the issues noted here of some concern.

It is surprising that the original IGS programs provide the best introduction to algebraic generality in a task such as the one described here. A perception that algebra is about symbols and graphical representation may have led to the implementation of these in more overtly algebraic programs with insufficient attention to facilitating student understanding at the most basic level of algebra.

There are a number of directions in which IGS could be developed. One direction might be to make it easier to create patterns to represent discrete variables. Another might be to expand the ways in which students can annotate and describe their work. Cinderella and GSP, for example, have a scribble tool, enabling arbitrary marks to be made on the screen, and Casyopée allows the user to compose a description of each object as it is constructed. Further options for calculation could be developed, perhaps to allow simple calculations on static numbers as well as the formulation of specific expressions. Another critical direction is the interface: many of the issues found here have to do with clear communication with the user rather than the presence or absence of certain features. See [11] for further discussion of this issue.

Interactive geometry software has great potential as a tool for facilitating algebraic understanding; hopefully this paper has indicated some of the ways in which the needs of learners with little algebraic awareness might be addressed in order that this potential might be more fully realized.

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